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# The *GW* approximation in less than 60 minutes

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## I. Standard DFT suffers from the band gap problem



II. Introduction of the Green's function

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## III. The GW approximation

IV. The GW code in ABINIT and the  $G_{_0}W_{_0}$  method

### V. Some applications

#### Standard DFT has unfortunately some shortcomings



#### A pervasive problem





FIG. 1. Single-particle Hartree-Fock and local density approximation eigenvalue spectra (eV) for the SiH<sub>4</sub> molecule.

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#### How do go beyond within the DFT framework?

Not easy to find improvement within DFT framework There is no such thing as a perturbative expansion Perdew's Jacob's ladder does not help for the band gap HEAVEN OF CHEMICAL ACCURACY unoccupied  $\{\phi_i\}$ generalized RPA hyper-GGA εx  $\tau$  and/or  $\nabla^2 n$ meta-GGA eneraie atomique • eneraies alterr after J. Perdew JCP (2005).  $\nabla n$ GGA LSD n HARTREE WORLD

FIG. 1. Jacob's ladder of density functional approximations to the exchange-correlation energy.

#### Need to change the overall framework!

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Many-body perturbation theory

Historically older than the DFT (1940-50's)! Big names: Feynman, Schwinger, Hubbard, Hedin, Lundqvist



The Green's function

Exact ground state wavefunction: |N , 0
angle

Creation, annihilation operator:  $\Psi^{\dagger}(\mathbf{r} t)$ ,  $\Psi(\mathbf{r} t)$ 

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• 
$$\Psi^{\dagger}(\mathbf{r}t)|N,0\rangle$$

is a (N+1) electron wavefunction not necessarily in the ground state

2  $\Psi^\dagger(r't')|N,0
angle$  is another (N+1) electron wavefunction

#### Let's compare the two of them!

#### Green's function definition



$$= i G^{e}(rt, r't')$$
 for  $t > t'$ 

#### Mesures how an extra electron propagates from (r't') to (rt).

#### Green's function definition



 $= i G^{h}(r't', rt)$  for t' > t

Mesures how a missing electron (= a hole) propagates from (rt) to (r't').

Final expression for the Green's function

$$i G(\mathbf{r}t, \mathbf{r}'t') = \langle N, \mathbf{0} | T [\Psi(\mathbf{r}t) \Psi^{\dagger}(\mathbf{r}'t')] | N, \mathbf{0} \rangle$$
  
time-ordering operator

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$$G(rt, r't') = G^{e}(rt, r't') -G^{h}(r't', rt)$$

Compact expression that describes both the propagation of an extra electron and an extra hole

#### Lehman representation

$$i G(\mathbf{r}, \mathbf{r}', t-t') = \langle N, 0 | T [\Psi(\mathbf{r}t) \Psi^{\dagger}(\mathbf{r}'t')] | N, 0 \rangle$$
Closure relation
$$\sum_{N,i} | N, i \rangle \langle N, i |$$
Lehman representation:
$$G(\mathbf{r}, \mathbf{r}', \omega) = \sum_{i} \frac{f_{i}(\mathbf{r}) f_{i}^{*}(\mathbf{r}')}{\omega - \epsilon_{i} \pm i \eta}$$
where
$$\epsilon_{i} = \begin{cases} E(N+1,i) - E(N,0) \\ E(N,0) - E(N-1,i) \end{cases}$$
Exact excitation energies!

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#### Related to photoemission spectroscopy



Energy conservation: before after  $h \nu + E(N,0) = E_{kin} + E(N-1,i)$ 

**Quasiparticle energy:**  $\epsilon_i = E(N,0) - E(N-1,i) = E_{kin} - hv$ 

#### And inverse photoemission spectroscopy



Energy conservation: before after  $E_{kin} + E(N,0) = hv + E(N+1,i)$ 

uasiparticle energy: 
$$\epsilon_i = E(N+1,i) - E(N,0) = E_{kin} - hv$$

#### Other properties of the Green's function

Galitskii-Migdal formula for the total energy:

$$E_{total} = \frac{1}{\pi} \int_{-\infty}^{\mu} d\omega \operatorname{Tr}\left[\left(\omega - h_{0}\right) \operatorname{Im} G\left(\omega\right)\right]$$



Expectation value of any 1 particle operator (local or non-local)  $\langle O \rangle = \lim_{t \to t'} Tr[OG]$ 

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How to calculate the Green's function?

#### Feynman diagrams

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#### Hedin's functional approach PRA (1965).

6 coupled equations:  $1 = (\mathbf{r_1} t_1 \sigma_1)$   $2 = (\mathbf{r_2} t_2 \sigma_2)$ 

→ 
$$G(1,2) = G_0(1,2) + \int d34 G_0(1,3) \Sigma(3,4) G(4,2)$$
 Dyson equation  
 $\Sigma(1,2) = i \int d34 G(1,3) W(1,4) \Gamma(4,2,3)$  self-energy  
 $\Gamma(1,2,3) = \delta(1,2) \delta(1,3) + \int d4567 \frac{\delta \Sigma(1,2)}{\delta G(4,5)} G(4,6) G(5,7) \Gamma(6,7,3)$  vertex  
 $X_0(1,2) = -i \int d34 G(1,3) G(4,1) \Gamma(3,4,2)$  polarizability  
 $\varepsilon(1,2) = \delta(1,2) - \int d3 v(1,3) X_0(3,2)$  dielectric matrix  
 $W(1,2) = \int d3 \varepsilon^{-1}(1,3) v(3,2)$  screened Coulomb interaction

Simplest approximation



Not enough: Hartree-Fock is know to be quite bad for solids

#### Hartree-Fock approximation for band gaps





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6 coupled equations:

$$G(1,2) = G_0(1,2) + \int d34 \, G_0(1,3) \, \Sigma(3,4) \, G(4,2)$$
Dyson equation
$$\Sigma(1,2) = i \int d34 \, G(1,3) \, W(1,4) \, \Gamma(4,2,3)$$
self-energy
$$\Gamma(1,2,3) = \delta(1,2) \, \delta(1,3) + \int d \, 4567 \frac{\delta \, \Sigma(1,2)}{\delta \, G(4,5)} \, G(4,6) \, G(5,7) \, \Gamma(6,7,3)$$

$$X_0(1,2) = -i \int d34 \, G(1,3) \, G(4,1) \, \Gamma(3,4,2)$$

$$\varepsilon(1,2) = \delta(1,2) - \int d3 \, v(1,3) \, X_0(3,2)$$
W(1,2) =  $\int d3 \, \varepsilon^{-1}(1,3) \, v(3,2)$ 
screened Coulomb interaction

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$$X_0(1,2) = -i \int d34 \, G(1,3) \, G(4,1) \, \Gamma(3,4,2)$$
  

$$\varepsilon(1,2) = \delta(1,2) - \int d3 \, v(1,3) \, X_0(3,2)$$
  

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 screened Coulomb interaction

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6 coupled equations:

$$G(1,2) = G_0(1,2) + \int d34 G_0(1,3) \Sigma(3,4) G(4,2)$$
Dyson equation
$$\Sigma(1,2) = i \int d34 G(1,2) W(1,2) \frac{\Gamma(4,2,3)}{\delta G(4,2)}$$
self-energy
$$\Gamma(1,2,3) = \delta(1,2) \delta(1,3) + \int d 4567 \frac{\delta \Sigma(1,2)}{\delta G(4,5)} G(4,6) G(5,7) \Gamma(6,7,3)$$

$$X_0(1,2) = -i \int d34 G(1,2) G(2,1) \frac{\Gamma(3,4,2)}{\delta G(4,5)}$$

$$\varepsilon(1,2) = \delta(1,2) - \int d3 v(1,3) X_0(3,2)$$

$$W(1,2) = \int d3 \varepsilon^{-1}(1,3) v(3,2)$$
screened Coulomb interaction

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#### Here comes the GW approximation

 $\Sigma(1,2) = i G(1,2) W(1,2)$ 

#### GW approximation



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$$\chi_0(1,2) = -i G(1,2) G(2,1)$$

 $\varepsilon(1,2) = \delta(1,2) - \int d3 v(1,3) X_0(3,2)$ 

 $W(1,2) = \int d3 \, \epsilon^{-1}(1,3) \, v(3,2)$ 

Interaction between electrons in vacuum:

$$V(\mathbf{r},\mathbf{r'}) = \frac{1}{4\pi\epsilon_0} \frac{e^2}{|\mathbf{r}-\mathbf{r'}|}$$



Interaction between electrons in a homogeneous polarizable medium:



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Dynamically screened interaction between electrons in a general medium:

$$W(\mathbf{r},\mathbf{r}',\omega) = \frac{e^2}{4\pi\epsilon_0} \int d\mathbf{r}'' \frac{\varepsilon^{-1}(\mathbf{r},\mathbf{r}'',\omega)}{|\mathbf{r}''-\mathbf{r}'|}$$



#### W is frequency dependent

W can measured directly by Inelastic X-ray Scattering



Zero below the band gap

H-C Weissker et al. PRB (2010)

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#### GW has a "super" Hartree-Fock





#### GW approximation gets good band gap



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#### Available GW codes

	P De B		New features   Log in / create acc				ate account
		Article Discussion	Read	Edit View his	tory	Search	Q
	WIKIPEDIA	GW approximation					
œ	Main page Contents Featured content Current events Random article	The <b>GW approximation</b> (GWA) is an approximation made in order to calculate the self-energy of a many-body system. The approximation is that the expansion of $G$ and the screened interaction $W$ $\Sigma = i GW - GWGWG + \cdots$ can be truncated after the first term: $\Sigma \approx i GW$	the self	-energy Σ in terr	ns of th	he single particle Green	function
	<ul> <li>Interaction</li> <li>About Wikipedia</li> <li>Community portal</li> <li>Recent changes</li> <li>Contact Wikipedia</li> <li>Donate to</li> <li>Wikipedia</li> <li>Help</li> </ul>	Another way to say the same thing is that that self-energy is expanded in a formal Taylor series in powers of the screened interaction <i>W</i> and the lowest order term is kept in the expansion in GWA. To put this in context, if one replaces <i>W</i> by the bare Coulomb interaction, one generates the usual series for the self-energy found in most many-body textbooks. The GWA with <i>W</i> replaced by the bare Coulomb yields nothing other than the Hartree-Fock exchange potential (self-energy). Therefore, loosely speaking, the GWA represents a type of dynamically screened Hartree-Fock self-energy. In a solid state system, the series for the self-energy in terms of <i>W</i> should converge much faster than the traditional series in the bare Coulomb interaction. This is because the screening of the medium reduces the effective strength of the Coulomb interaction: for example, if one places an electron at some position in a material and asks what the potential is at some other position in the material, the value is smaller than given by the bare Coulomb interaction go that a series in <i>W</i> should have higher hopes of converging quickly.					
	▶ Toolbox	Software supporting the GW approximation					[edit]
	<ul> <li>Print/export</li> <li>Languages Italiano 日本語 中文</li> </ul>	ABINIT - plane wave pseudopotential method     Spex @ - full-potential (linearized) augmented plane-wave (FP-LAPW) method     SaX @ - plane wave pseudopotential method     YAMBO code- plane wave pseudopotential method					
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- L. Hedin, Phys. Rev. 139, A796 (1965).
- W.G. Aulbur, L. Jönsson and J.W. Wilkins, Solid State Physics 54, 1 (2000) full version in postscript (7,3MB) 🖉.
- F. Aryasetiawan, O. Gunnarsson, arXiv:cond-mat/9712013v1@.

#### Available GW codes



has a GW code inside

#### Code history



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How to get *G*?

Remember the Lehman representation:  

$$G(\mathbf{r}, \mathbf{r}', \omega) = \sum_{i} \frac{f_i(\mathbf{r}) f_i^*(\mathbf{r}')}{\omega - \epsilon_i \pm i\eta}$$
where the  $f_i(\mathbf{r})$  and the  $\epsilon_i$  are complicated quantities

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But for independent electrons like Kohn-Sham electrons:

$$G^{KS}(\mathbf{r},\mathbf{r}',\omega) = \sum_{i} \frac{\varphi_{i}^{KS}(\mathbf{r})\varphi_{i}^{KS*}(\mathbf{r}')}{\omega - \epsilon_{i}^{KS} \pm i\eta}$$

This can be considered as the best guess for G

One can get W and  $\Sigma^{GW}$ 

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#### *GW* as a perturbation with respect to LDA

GW quasiparticle equation:

$$\left[h_{0}+\Sigma_{xc}(\epsilon_{i}^{GW})\right]\left|\varphi_{i}^{GW}\right\rangle=\epsilon_{i}^{GW}\left|\varphi_{i}^{GW}\right\rangle$$



#### *GW* as a perturbation with respect to LDA

GW quasiparticle equation:

KS equation:

$$\langle \varphi_i^{\text{LDA}} | [h_0 + \Sigma_{xc}(\epsilon_i^{GW})] | \varphi_i^{\text{LDA}} \rangle = \epsilon_i^{GW}$$



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$$\langle \varphi_i^{\text{LDA}} | [h_0 + v_{xc}^{\text{LDA}}] | \varphi_i^{\text{LDA}} \rangle = \epsilon_i^{\text{LDA}}$$

$$\epsilon_{i}^{GW} - \epsilon_{i}^{\text{LDA}} = \left\langle \varphi_{i}^{\text{LDA}} \middle| \left[ \Sigma_{xc}(\epsilon_{i}^{GW}) - v_{xc}^{\text{LDA}} \right] \middle| \varphi_{i}^{\text{LDA}} \right\rangle$$

Linearization of the energy dependance

$$\boldsymbol{\epsilon}_{i}^{GW} - \boldsymbol{\epsilon}_{i}^{\text{LDA}} = \left\langle \boldsymbol{\varphi}_{i}^{\text{LDA}} \middle| \left[ \boldsymbol{\Sigma}_{xc} \left( \boldsymbol{\epsilon}_{i}^{GW} \right) - \boldsymbol{v}_{xc}^{\text{LDA}} \right] \middle| \boldsymbol{\varphi}_{i}^{\text{LDA}} \right\rangle$$

Not yet known

Taylor expansion:  

$$\underbrace{\square \Sigma_{xc}(\epsilon_i^{GW}) = \sum_{xc}(\epsilon_i^{\text{LDA}}) + (\epsilon_i^{GW} - \epsilon_i^{\text{LDA}}) \frac{\partial \Sigma_{xc}}{\partial \epsilon} + \dots}{\partial \epsilon}$$

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#### **Final result:**

$$\boldsymbol{\epsilon}_{i}^{GW} = \boldsymbol{\epsilon}_{i}^{\text{LDA}} + Z_{i} \left\langle \boldsymbol{\varphi}_{i}^{\text{LDA}} \middle\| \left[ \boldsymbol{\Sigma}_{xc} (\boldsymbol{\epsilon}_{i}^{\text{LDA}}) - \boldsymbol{v}_{xc}^{\text{LDA}} \right] \middle| \boldsymbol{\varphi}_{i}^{\text{LDA}} \right\rangle$$

where 
$$Z_i = 1/\left(1 - \frac{\partial \Sigma_{xc}}{\partial \epsilon}\right)$$

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#### A typical ABINIT ouptput for Silicon at Gamma point



#### Flow chart of a typical GW calculation



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#### GW approximation gets good band gap



#### Clusters de sodium



1<sup>st</sup> Yarmouk school, Irbid 4 november 2010

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#### Calculs de défauts avec l'approximation GW

Calcul d'un système à 215 atomes



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Carbure de Silicium cubique 3C-SiC

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#### Band Offset at the interface between two semiconductors



GW correction with respect to LDA

R. Shaltaf PRL (2008).

Summary

- The GW approximation solves the band gap problem!
- The calculations are extremely heavy, so that we resort to many additional technical approximations: method named G<sub>0</sub>W<sub>0</sub>
- The complexity comes from
  - Dependance upon empty states
  - Non-local operators
  - Dynamic operators that requires freq. convolutions
- There are still some other approximations like the Plasmon-Pole model... that I'll discuss during the practical session...